Kuwait University Math 101 Date: October 25, 2001
Dept of Maths. & Comp. Sci. First Exam. Duration: 75 minutes

Calculators and Mobile Phones are not allowed

1. **(9 points)** Find the following limits, if they exist

(a) $\lim_{x\to 1} \dfrac{x-1}{\sqrt[3]{x}-1}$

(b) $\lim_{x\to 2} |x-2| \sin\left(\dfrac{\pi}{x-2}\right)$

(c) $\lim_{x\to 0} \dfrac{1-\cos(3x)}{\sin^2(x)}$

2. **(4 points)** Find the vertical and horizontal asymptotes, if any of

$$f(x) = \frac{2x+\sqrt{x^2+1}}{x+2}$$

3. **(4 points)** State the Intermediate Value Theorem. Then use it to show that the graphs of f and g intersect, where

$$f(x) = x^3 + 7x^2 - 5 \text{ and } g(x) = 5 - x - 2x^2 \cdot$$

4. **(4 points)** Write the definition of the derivative of f at 1. Then use it to find $f'(1)$ if it exists, where

$$f(x) = \sqrt{x^2 - 2x + 1} \cdot$$

Justify your answer.

5. **(4 points)** Find the points on the graph of f at which the tangent line is perpendicular to the line $7y + x = 1$, where

$$f(x) = x^3 + x^2 + 6x - 5 \cdot$$

$$\frac{x-1}{\sqrt[3]{x}-1} = x^{2/3} + x^{1/3} + 1 \Longrightarrow 3 \text{ when } x \Longrightarrow 1$$

For (b), we have

$$-|x-2| \leq |x-2| \sin\left(\frac{\pi}{x-2}\right) \leq |x-2|$$

the Sandwich theorem will then imply

$$|x-2| \sin\left(\frac{\pi}{x-2}\right) \Longrightarrow 0 \text{ when } x \Longrightarrow 2$$

For (c), we have

$$\lim_{x \to 0} \frac{1-\cos 3x}{\sin^2 x} \frac{1+\cos 3x}{1+\cos 3x} = \lim_{x \to 0} \left(\frac{\sin^2 3x}{9x^2}\right)\left(\frac{x^2}{\sin^2 x}\right)\left(\frac{9}{1+\cos 3x}\right) = \frac{9}{2}$$

Problem 2. We have

$$\frac{2x+\sqrt{x^2+1}}{x+2} = \frac{2x+|x|\sqrt{1+1/x^2}}{x(1+2/x)}$$

so when $x > 0$, we have $f(x) = \dfrac{2+\sqrt{1+1/x^2}}{1+2/x}$ which implies

$$\lim_{x \to \infty} f(x) = 3 \text{ that is } y = 3 \text{ is H.A.}$$

and if $x < 0$, we have $f(x) = \dfrac{2-\sqrt{1+1/x^2}}{1+2/x}$ which implies

$$\lim_{x \to \infty} f(x) = 1 \text{ that is } y = 1 \text{ is H.A.}$$

Since denominator is zero at $x = -2$, and since

$$\lim_{x \to -2^-} f(x) = +\infty, \lim_{x \to -2^+} f(x) = \infty \Rightarrow x = -2 \text{ is V.A.}$$

Problem 3. The graphs of f and g intersect iff there exists x such that $f(x) = g(x)$ iff there exists x such that $f(x) - g(x) = 0$. Set

$$h(x) = f(x) - g(x) = x^3 + 9x^2 + x - 10$$

Since $h(0) = -10$ and $h(1) = 1$, and f is continuous on $[0, 1]$, the Intermediate Value Theorem implies the existence of $x \in]0, 1[$ such that $h(x) = 0$.

Problem 4. The definition is

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x-1}$$

For $f(x) = \sqrt{x^2 - 2x + 1}$, we get

$$\frac{f(x)-f(1)}{x-1} = \frac{\sqrt{x^2-2x+1}}{x-1} = \frac{\sqrt{(x-1)^2}}{x-1} = \frac{|x-1|}{x-1}$$

and since $\lim\limits_{x \to 1} \dfrac{|x-1|}{x-1}$ does not exist (because it is easy to check that the right-limit is 1 and the left-limit is -1), so $f'(1)$ does not exist.

Problem 5. The slope of the line $7y + x = 1$ is $-1/7$. Any normal line to this line will have slope 7. Since any tangent line to $f(x)$ has slope $f'(x) = 3x^2 + 2x + 6$, we must have

$$= 7 \Rightarrow 3x^2 + 2x - 1 = 0$$